A note on a model of inequality aversion and public goods when the marginal rate of substitution is not constant

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Abstract

This paper is a supplementary material of Yokoo (2018) "A model of inequality aversion and public goods when the marginal rate of substitution is not constant."

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Model and optimal response

Consider an economy consisting two individuals, individual A and B. Assume that there are only a private and a public good. Each individual i consumes an amount x_i of the private good and donates an amount g_i to the supply of the public good. Let $G = g_A + g_B$ be the total private contributions to the public good. Both individuals i are endowed with wealth $w > 0$, which they allocate between private good x_i and contribution g_i. Let $\pi_i = (x_i, G)$ be individual i's sub-utility, which corresponds to monetary payoffs in the model of Fehr and Schmidt (1999). Assume that $\frac{\partial \pi}{\partial x} \geq 0$, $\frac{\partial^2 \pi}{\partial x^2} \leq 0$, $\frac{\partial \pi}{\partial G} \geq 0$, and $\frac{\partial^2 \pi}{\partial G^2} \leq 0$.

Following Fehr and Schmidt (1999), consider individual B's preferences as:

$$U_B = \begin{cases} \pi_B - \alpha\left(\pi_A - \pi_B\right), \text{ if } \pi_B \leq \pi_A \\ \pi_B - \beta\left(\pi_B - \pi_A\right), \text{ if } \pi_B > \pi_A. \end{cases} \tag{1}$$

For guilt parameter β, Fehr and Schmidt (1999) assume $0 \leq \beta < 1$. This assumption is also made in this study. For the envy parameter α, assume that $0 \leq \alpha$.[1] In Yokoo (2018), the Cobb-Douglas function of $\pi\left(x_i, G\right) = \gamma \log x_i + (1-\gamma)\log G$, where $\gamma \in (0,1)$ is examined for the case when MRS is not constant. Then, individual B's contribution g_B can be found by solving

$$\max_{x_B, g_B} U_B = \begin{cases} -\alpha\left[\gamma \log\ x_A + (1-\gamma)\log G\right] + (1+\alpha)\left[\gamma \log x_B + (1-\gamma)\log G\right], \\ \text{if } x_B \leq x_A \\ \beta\left[\gamma \log\ x_A + (1-\gamma)\log G\right] + (1-\beta)\left[\gamma \log x_B + (1-\gamma)\log G\right], \\ \text{if } x_B > x_A. \end{cases} \tag{2}$$

[1] Fehr and Schmidt (1999) further assume $\beta \leq \alpha$.

$$\text{s.t. } x_A + g_A = w,\ x_B + g_B = w,\ g_A + g_B = G.$$

Following Bergstrom et al. (1986), it is assumed that individual B takes the contribution of A as exogenously given (the Nash assumption). Also assume $g_A \in (0, w)$. By substituting $g_B = G - g_A$ into the above and the budget constraints into the utility function, the optimization problem is equivalent to

$$\max_G U_B = \begin{cases} -\alpha\gamma \log \ (w - g_A) + (1+\alpha)\gamma \log (w - G + g_A) + (1-\gamma)\log G, \\ \qquad \text{if } 2g_A \leq G \\ \beta\gamma \log \ (w - g_A) + (1-\beta)\gamma \log (w - G + g_A) + (1-\gamma)\log G, \\ \qquad \text{if } 2g_A > G. \end{cases} \tag{3}$$

This utility function is not differentiable. Therefore, to solve this maximization problem, the problem is splitted into two by adding the conditions $G - 2g_A \geq 0$ and $2g_A - G > 0$ as constraints. This makes U_B differentiable within each sub-problem. Then, each sub-problem can be solved by applying the Kuhn-Tucker conditions, and both interior and corner solutions to the original problem can be obtained by comparing the utility levels of the solutions to the two sub-problems. Using this algorithm, individual B's optimal response g_B^* is shown and studied in the next section.

Note that

$$MRS = \frac{\partial U_B/\partial x_B}{\partial U_B/\partial G} = \frac{(1-k)\,\gamma}{1-\gamma} . \frac{G}{w+g_A-G},$$

where $k = -\alpha, \beta$, which means the MRS between the private and public goods is not constant. Further, the MRS when approximating from the left and right to $G = 2g_A$ differ if $\alpha \neq \beta$.

To derive optimal response $g_B^*(g_A)$, the problem with constraint $G - 2g_A \geq 0$ is examined. This is the case where individual B's sub-utility is relatively low or equal to that of A:

$$\max_G U_B = -\alpha\gamma \log(w-g_A) + (1+\alpha)\,\gamma \log(w-G+g_A) + (1-\gamma)\log G, \quad (4)$$

$$\text{s.t. } G - 2g_A \geq 0.$$

Then the *Lagrangian function* can be defined as

$$\begin{aligned} L(G) = &-\alpha\gamma \log\ (w-g_A) + (1+\alpha)\,\gamma \log(w-G+g_A) + (1-\gamma)\log G \\ &+ \lambda_1 (G - 2g_A)\,, \end{aligned} \quad (5)$$

where λ_1 is the Lagrangian multiplier. Then the Kuhn-Tucker conditions of the problem are

$$\begin{aligned}
\frac{\partial L(G)}{\partial G} &= -\frac{(1+\alpha)\gamma}{w-G+g_A}+\frac{1-\gamma}{G}+\lambda_1=0\\
\frac{\partial L(G)}{\partial \lambda_1} &= G-2g_A\geq 0\\
\lambda_1\cdot\frac{\partial L(G)}{\partial \lambda_1} &= \lambda_1(G-2g_A)=0\\
\lambda_1 &\geq 0.
\end{aligned}$$

The solutions for this sub-problem are

$$g_B=\begin{cases}\frac{1-\gamma}{1+\alpha\gamma}w-\frac{(1+\alpha)\gamma}{1+\alpha\gamma}g_A, & \text{if } 0<g_A<g_A^{\alpha}\\ g_A, & \text{if } g_A^{\alpha}\leq g_A,\end{cases} \tag{6}$$

where $g_A^{\alpha}=\frac{1-\gamma}{1+\gamma+2\alpha\gamma}w$.

Next, examine the problem with the constraint of $2g_A-G>0$, which is the case where individual B's subutility is relatively high:

$$\max_G U_B=\beta\gamma\log\ (w-g_A)+(1-\beta)\gamma\log(w-G+g_A)+(1-\gamma)\log G, \tag{7}$$

$$\text{s.t. } 0<g_A\leq G\leq 2g_A.$$

where $0<\beta<1, w>0$. To make a problem solvable, we change $G<2g_A$

to $G \leq 2g_A$. Then the Lagrangian function can be defined as

$$\begin{aligned}L(G) &= \beta\gamma \log\ (w - g_A) + (1-\beta)\gamma \log(w - G + g_A) + (1-\gamma)\log G \\ &\quad + \lambda_2 [G - g_A] + \lambda_3 [2g_A - G] \qquad (8)\end{aligned}$$

Then the Kuhn-Tucker conditions of the problem are

$$\begin{aligned}\frac{\partial L(G)}{\partial G} &= -\frac{(1-\beta)\gamma}{w - G + g_A} + \frac{1-\gamma}{G} + \lambda_2 - \lambda_3 = 0 \\ \frac{\partial L(G)}{\partial \lambda_2} &= G - g_A \geq 0 \\ \frac{\partial L(G)}{\partial \lambda_3} &= 2g_A - G \geq 0 \\ \lambda_2 \cdot \frac{\partial L(G)}{\partial \lambda_2} &= \lambda_2 [G - g_A] = 0 \\ \lambda_3 \cdot \frac{\partial L(G)}{\partial \lambda_3} &= \lambda_3 [2g_A - G] = 0 \\ \lambda_2 \geq 0, \lambda_3 &\geq 0.\end{aligned}$$

For $0 \leq \beta < 1 - \frac{1-\gamma}{\gamma}$, the solutions for this sub-problem are

$$g_B = \begin{cases} g_A & \text{if } 0 < g_A \leq \underline{g_A^\beta} \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A, & \text{if } \underline{g_A^\beta} < g_A \leq \overline{g_A^\beta} \\ 0, & \text{if } \overline{g_A^\beta} \leq g_A < w, \end{cases} \qquad (9)$$

where $\underline{g_A^\beta} = \frac{1-\gamma}{1+\gamma-2\beta\gamma} w$ and $\overline{g_A^\beta} = \frac{1-\gamma}{(1-\beta)\gamma} w$. For $1 - \frac{1-\gamma}{\gamma} \leq \beta$, the solutions for this sub-problem is

$$g_B = \begin{cases} g_A & \text{if } 0 < g_A \leq \underline{g_A^\beta} \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A, & \text{if } \underline{g_A^\beta} < g_A < w. \end{cases} \tag{10}$$

If $g_B = g_A$, then

$$U_B = \gamma \log (w - g_A) + (1 - \gamma) \log 2 g_A. \tag{11}$$

This means that the level of U_B is independent of α and β. Note that, from the first sub-problem, $g_B = \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A$ gives us higher utility than (11) for $0 < g_A < g_A^\alpha$. A same argument applies for $g_B = \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A$ and 0, for $0 < g_A \leq \underline{g_A^\beta}$ and $\overline{g_A^\beta} \leq g_A < w$ for the second sub-problem, respectly.

As a result, the solutions of the two sub-problems are compared to derive the optimal response:

$$g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A, & \text{for } 0 < g_A < g_A^\alpha \\ g_A, & \text{for } g_A^\alpha \leq g_A \leq \underline{g_A^\beta} \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A, & \text{for } \underline{g_A^\beta} < g_A \leq \overline{g_A^\beta} \\ 0, & \text{for } \overline{g_A^\beta} \leq g_A < w, \end{cases} \tag{12}$$

when $0 \leq \beta < 1 - \frac{1-\gamma}{\gamma}$. When $1 - \frac{1-\gamma}{\gamma} \leq \beta$,

$$
g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A, & \text{for } 0 < g_A < g_A^\alpha \\ g_A, & \text{for } g_A^\alpha \leq g_A \leq \underline{g_A^\beta} \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A, & \text{for } \underline{g_A^\beta} < g_A < w. \end{cases} \tag{13}
$$

References

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