Inequality aversion and the private provision of public goods when the marginal rate of substitution is not constant

Online Appendix

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Model and optimal response

Consider a model where there is one private good, one public good and two individuals, A and B. Each individual i consumes an amount x_i of the private good and donates an amount g_i to the supply of the public good. Let $G = g_A + g_B$ be the total private contributions to the public good. Both individuals i are endowed with wealth $w > 0$, which they allocate between private good x_i and contribution g_i. Let $\pi_i = \pi(x_i, G)$ be individual i's sub-utility, which corresponds to monetary payoffs in the model of Fehr and Schmidt (1999). Assume that $\frac{\partial \pi}{\partial x} \geq 0$, $\frac{\partial^2 \pi}{\partial x^2} \leq 0$, $\frac{\partial \pi}{\partial G} \geq 0$, and $\frac{\partial^2 \pi}{\partial G^2} \leq 0$.

Following Fehr and Schmidt (1999), consider individual B's preferences as:

$$U_B = \begin{cases} \pi_B - \alpha(\pi_A - \pi_B) & \text{if } \pi_B \leq \pi_A, \\ \pi_B - \beta(\pi_B - \pi_A) & \text{if } \pi_B > \pi_A. \end{cases} \tag{1}$$

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For guilt parameter β, Fehr and Schmidt (1999) assume $0 \le \beta < 1$. This assumption is also made in this study. For the envy parameter α, assume that $0 \le \alpha$.[1] In this study, the Cobb-Douglas function of $\pi(x_i, G) = \gamma \log x_i + (1-\gamma)\log G$, where $\gamma \in (0,1)$ is examined for the case when MRS is not constant. Then, individual B's contribution g_B can be found by solving

$$\max_{x_B, g_B} U_B = \begin{cases} -\alpha\left[\gamma \log\ x_A + (1-\gamma)\log G\right] + (1+\alpha)\left[\gamma \log x_B + (1-\gamma)\log G\right] \\ \text{if } x_B \le x_A, \\ \beta\left[\gamma \log\ x_A + (1-\gamma)\log G\right] + (1-\beta)\left[\gamma \log x_B + (1-\gamma)\log G\right] \\ \text{if } x_B > x_A. \end{cases} \tag{2}$$

$$\text{s.t. } x_A + g_A = w,\ x_B + g_B = w,\ g_A + g_B = G.$$

Following Bergstrom et al. (1986), it is assumed that individual B takes the contribution of A as exogenously given (the Nash assumption). Also assume $g_A \in (0, w)$. By substituting $g_B = G - g_A$ into the above and the budget constraints into the utility function, the optimization problem is equivalent to

$$\max_{G} U_B = \begin{cases} -\alpha\gamma \log\ (w - g_A) + (1+\alpha)\gamma \log(w - G + g_A) + (1-\gamma)\log G \\ \text{and } 2g_A \le G, \\ \beta\gamma \log\ (w - g_A) + (1-\beta)\gamma \log(w - G + g_A) + (1-\gamma)\log G \\ \text{and } 2g_A > G. \end{cases} \tag{3}$$

This utility function is not differentiable if $\alpha \ne 0$ or $\beta \ne 0$. Therefore, to solve this

[1] Fehr and Schmidt (1999) further assume $\beta \le \alpha$.

maximization problem, the problem is splitted into two by adding the conditions $G-2g_A \geq 0$ and $2g_A-G>0$ as constraints. This makes U_B differentiable within each sub-problem. Then, each sub-problem can be solved by applying the Kuhn-Tucker conditions, and both interior and corner solutions to the original problem can be obtained by comparing the utility levels of the solutions to the two sub-problems.

Note that

$$MRS = \frac{\partial U_B/\partial x_B}{\partial U_B/\partial G} = \frac{(1-k)\,\gamma}{1-\gamma}.\frac{G}{w+g_A-G},$$

where $k=-\alpha,\beta$, which means the MRS between the private and public goods is not constant. Further, the MRS when approximating from the left and right to $G=2g_A$ differ if $-\alpha \neq \beta$.

To derive optimal response $g_B^*\left(g_A\right)$, the problem with constraint $2g_A \leq G$ is examined. This is the case where individual B's sub-utility is relatively low or equal to that of A:

$$\max_G U_B = -\alpha\gamma \log\left(w-g_A\right) + \left(1+\alpha\right)\gamma \log\left(w-G+g_A\right) + \left(1-\gamma\right)\log G, \tag{4}$$

$$\text{s.t. } G-2g_A \geq 0.$$

Then the *Lagrangian function* can be defined as

$$\begin{aligned} L\left(G\right) = & -\alpha\gamma \log \ \left(w-g_A\right) + \left(1+\alpha\right)\gamma \log\left(w-G+g_A\right) + \left(1-\gamma\right)\log G \\ & + \lambda_1\left(G-2g_A\right), \end{aligned} \tag{5}$$

where λ_1 is the Lagrangian multiplier. Then the Kuhn-Tucker conditions of the problem are

$$
\begin{aligned}
\frac{\partial L(G)}{\partial G} &= -\frac{(1+\alpha)\gamma}{w-G+g_A} + \frac{1-\gamma}{G} + \lambda_1 = 0 \\
\frac{\partial L(G)}{\partial \lambda_1} &= G - 2g_A \geq 0 \\
\lambda_1 \cdot \frac{\partial L(G)}{\partial \lambda_1} &= \lambda_1 (G - 2g_A) = 0 \\
\lambda_1 &\geq 0.
\end{aligned}
$$

The solution for this sub-problem if

$$
g_B = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A & \text{for } 0 < g_A < g_A^{\alpha}, \\ g_A & \text{for } g_A^{\alpha} \leq g_A < w, \end{cases} \tag{6}
$$

where $g_A^{\alpha} = \frac{1-\gamma}{1+\gamma+2\alpha\gamma} w$.

Next, examine the problem with the constraint of $g_A \leq G < 2g_A$, which is the case where individual B's sub-utility is relatively high. To make a problem solvable, we change $G < 2g_A$ to $G \leq 2g_A$:

$$
\max_G U_B = \beta\gamma \log \ (w - g_A) + (1-\beta)\gamma \log (w - G + g_A) + (1-\gamma) \log G, \tag{7}
$$

$$
\text{s.t. } g_A \leq G \leq 2g_A.
$$

Then the Lagrangian function can be defined as

$$L(G) = \beta\gamma \log (w - g_A) + (1-\beta)\gamma \log(w - G + g_A) + (1-\gamma)\log G$$
$$+ \lambda_2 (G - g_A) + \lambda_3 (2g_A - G) \tag{8}$$

Then the Kuhn-Tucker conditions of the problem are

$$\begin{aligned}
\frac{\partial L(G)}{\partial G} &= -\frac{(1-\beta)\gamma}{w - G + g_A} + \frac{1-\gamma}{G} + \lambda_2 - \lambda_3 = 0 \\
\frac{\partial L(G)}{\partial \lambda_2} &= G - g_A \geq 0 \\
\frac{\partial L(G)}{\partial \lambda_3} &= 2g_A - G \geq 0 \\
\lambda_2 \cdot \frac{\partial L(G)}{\partial \lambda_2} &= \lambda_2 (G - g_A) = 0 \\
\lambda_3 \cdot \frac{\partial L(G)}{\partial \lambda_3} &= \lambda_3 (2g_A - G) = 0 \\
\lambda_2 \geq 0, \lambda_3 &\geq 0.
\end{aligned}$$

Denote the level of U_B when $g_B = g_A$ as

$$U_{B,g_A} = \gamma \log (w - g_A) + (1-\gamma)\log 2g_A. \tag{9}$$

This means that the level of U_B is independent of α and β when $g_B = g_A$. Denote the level of U_B when $g_B = 0$ as

$$U_{B,0} = \beta\gamma \log (w - g_A) + (1-\beta)\gamma \log w + (1-\gamma)\log g_A. \tag{10}$$

Consider a case where $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$. Suppose that U_{B,g_A} is larger than $U_{B,0}$:

$$
\begin{aligned}
U_{B,g_A} &> U_{B,0} \\
&\leftrightarrow (1-\beta)\,\gamma \log \frac{w-g_A}{w} > -(1-\gamma)\log 2 \\
&\leftrightarrow \log \frac{w}{w-g_A} < \log 2^{\delta} \leftrightarrow \frac{w}{w-g_A} < 2^{\delta} \\
&\leftrightarrow g_A < \frac{2^{\delta}-1}{2^{\delta}} w
\end{aligned}
$$

where $\delta = \frac{1-\gamma}{(1-\beta)\gamma}$. Note that $0 < \delta < 1$ for $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$ and $0 < \gamma < 1$.

First, from the above, $U_{B,g_A} > U_{B,0}$ for $g_A \leq \frac{1-\gamma}{1+\gamma-2\beta\gamma} w$ since $\frac{1-\gamma}{1+\gamma-2\beta\gamma} = \frac{\delta}{2+\delta} < \frac{2^{\delta}-1}{2^{\delta}}$ for $0 < \delta < 1$. Second, in constrast to the above, $U_{B,g_A} \leq U_{B,0}$ for $\frac{1-\gamma}{(1-\beta)\gamma} w \leq g_A < w$ since $\frac{2^{\delta}-1}{2^{\delta}} < \delta$ for $0 < \delta < 1$.

To sum up, for $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$, the solution for this sub-problem is

$$
g_B = \begin{cases} g_A & \text{for } 0 < g_A \leq \underline{g_A^{\beta}}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g_A^{\beta}} < g_A < \overline{g_A^{\beta}}, \\ 0 & \text{for } \overline{g_A^{\beta}} \leq g_A < w, \end{cases} \tag{11}
$$

where $\underline{g_A^{\beta}} = \frac{1-\gamma}{1+\gamma-2\beta\gamma} w$ and $\overline{g_A^{\beta}} = \frac{1-\gamma}{(1-\beta)\gamma} w$.

For $1 - \frac{1-\gamma}{\gamma} \leq \beta$, the solutions for this sub-problem is

$$
g_B = \begin{cases} g_A & \text{for } 0 < g_A \leq \underline{g_A^{\beta}}, \\ \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A & \text{for } \underline{g_A^{\beta}} < g_A < w. \end{cases} \tag{12}
$$

Note that, from the first sub-problem, $g_B = \frac{1-\gamma}{1+\alpha\gamma} w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma} g_A$ gives higher utility than (9) for $0 < g_A < g_A^{\alpha}$. A same argument applies for $g_B = \frac{1-\gamma}{1-\beta\gamma} w - \frac{(1-\beta)\gamma}{1-\beta\gamma} g_A$ for $0 < g_A \leq \underline{g_A^{\beta}}$, and $g_B = 0$ for $\overline{g_A^{\beta}} \leq g_A < w$ for the second sub-problem. By using these arguments, the solutions of the two sub-problems are compared to derive the optimal response:

$$g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma}w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma}g_A & \text{for } 0 < g_A < g_A^\alpha, \\ g_A & \text{for } g_A^\alpha \leq g_A \leq \underline{g_A^\beta}, \\ \frac{1-\gamma}{1-\beta\gamma}w - \frac{(1-\beta)\gamma}{1-\beta\gamma}g_A & \text{for } \underline{g_A^\beta} < g_A < \overline{g_A^\beta}, \\ 0 & \text{for } \overline{g_A^\beta} \leq g_A < w. \end{cases} \tag{13}$$

when $0 < \beta < 1 - \frac{1-\gamma}{\gamma}$. When $1 - \frac{1-\gamma}{\gamma} \leq \beta$,

$$g_B^* = \begin{cases} \frac{1-\gamma}{1+\alpha\gamma}w - \frac{(1+\alpha)\gamma}{1+\alpha\gamma}g_A & \text{for } 0 < g_A < g_A^\alpha, \\ g_A & \text{for } g_A^\alpha \leq g_A \leq \underline{g_A^\beta}, \\ \frac{1-\gamma}{1-\beta\gamma}w - \frac{(1-\beta)\gamma}{1-\beta\gamma}g_A & \text{for } \underline{g_A^\beta} < g_A < w. \end{cases} \tag{14}$$

References

[1] Bergstrom, T., Blume, L. and Varian, H., 1986. On the private provision of public goods. Journal of Public Economics, 29(1), pp.25-49.

[2] Fehr, E. and Schmidt, K.M., 1999. A theory of fairness, competition, and cooperation. Quarterly Journal of Economics, 114(3), pp.817-868.